Journal of Approximation Theory **118**, 202–224 (2002)
doi:10.1006/jath.2002.3717

On a Conjecture of Ditzian and Runovskii[1]

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Communicated by Zeev Ditzian

Received July 9, 2001; accepted in revised form June 26, 2002

Let M_θ be the mean operator on the unit sphere in $\mathbb{R}^n$, $n \geqslant 3$, which is an analogue of the Steklov operator for functions of single variable. Denote by D the Laplace–Beltrami operator on the sphere which is an analogue of second derivative for functions of single variable. Ditzian and Runovskii have a conjecture on the norm of the operator $\theta^2 D(M_\theta)^m, m \geqslant 2$ from $X = L^p$ $(1 \leqslant p \leqslant \infty)$ to itself which can be expressed as

$$\lim_{m\to\infty} \sup\{||\theta^2 D(M_\theta)^m||_{(X,X)} : \theta \in (0,\pi)\} = 0.$$

We give a proof of this conjecture.

Key Words: ultraspherical polynomials; spherical harmonics; mean operator.

1. INTRODUCTION

Suppose $n \in \mathbb{N}$ and $n \geqslant 3$. Let $\Omega_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$. For a function $f \in L(\Omega_n)$ define

$$M_\theta(f)(x) = \frac{1}{D(\theta)} \int_{D(x,\theta)} f(y)\, dy, \qquad x \in \Omega_n,$$

where $D(x, \theta) = \{y \in \Omega_n : x \cdot y > \cos\theta\}$, $\theta \in (0, \pi)$ and

$$D(\theta) = |D(x,\theta)| = |\Omega_{n-1}| \int_0^\theta \sin^{n-2} t\, dt.$$

We call M_θ the mean operator on the sphere.

[1]Project supported by NNSF of China under the Grant 10071007.
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0021-9045/02 $35.00

For a function $f \in L(\Omega_n)$, its Fourier–Laplace expansion is

$$\sum_{k=0}^{\infty} Y_k(f),$$

where $Y_k(f)$ is the "projection" of f on the space $\mathscr{H}_k^n$ of spherical harmonics of degree k. Actually,

$$Y_k(f)(x) = c_{nk}(f * P_k^n)(x) = c_{nk} \int_{\Omega_n} f(y) P_k^n(xy)\, dy,$$

where

$$c_{nk} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \frac{(n+2k-2)(n+k-2)!}{(n+k-2)k!(n-2)!}$$

and $P_k^n(t)$ are the normalized ultraspherical polynomials

$$P_k^n(t) = \frac{P_k^{(\frac{n-3}{2},\frac{n-3}{2})}(t)}{P_k^{(\frac{n-3}{2},\frac{n-3}{2})}(1)} = \frac{\Gamma(\frac{n-1}{2})\Gamma(k+1)}{\Gamma(k+\frac{n-1}{2})} P_k^{(\frac{n-3}{2},\frac{n-3}{2})}(t) \qquad (-1 \leqslant t \leqslant 1),$$

where $P_k^{(\alpha,\beta)}$ are the Jacobi polynomials.

If $f, g \in L(\Omega_n)$ and g has the expansion

$$\sum_{k=1}^{\infty} -k(k+n-2)\, Y_k(f),$$

then we call g the second derivative of f and denote it by $D(f)$, where D means the Laplace–Beltrami operator.

For these basic concepts we refer to [WL]. The concept of spherical harmonics can also be found in [SW, Chap. 4, Sect. 2]. The Laplace–Beltrami operator is called the "spherical Laplacean" in [St, Chap. 3, Sect. 3]. For Jacobi polynomials (including ultraspherical polynomials) very detailed materials can be found in [Sz, Chaps. 4, 7, and Sect. 7.32, 8].

Ditzian and Runovskii [DR] proposed a conjecture which can be stated as follows: Let $m \in \mathbb{N}, m > 2$, then

$$\lim_{m\to\infty} \sup\{||D(M_\theta)^m||_{(X,X)}\theta^2 : \theta \in (0,\pi)\} = 0, \tag{1}$$

where $||\cdot||_{(X,X)}$ denotes the operator norm from $X = L^p(\Omega_n)$ $(1 \leqslant p \leqslant \infty)$ to itself.

Our purpose is to prove this conjecture. Define

$$T_m(\theta) = \theta^2 D(M_\theta)^m \qquad (m > 2).$$

We rewrite (1) as the following theorem.

THEOREM. *For* $X = L^p(\Omega_n)$ $(1 \leqslant p \leqslant \infty)$

$$\lim_{m\to\infty} \sup\{||T_m(\theta)||_{(X,X)} : \theta \in (0, \pi)\} = 0.$$

2. PRELIMINARIES

Let

$$h_\theta(t) = \frac{1}{D(\theta)} \chi_{(0,\theta)}(\arccos(t)), \qquad t \in (-1, 1).$$

We see that M_θ is a convolution operator with kernel h_θ, i.e.

$$M_\theta(f)(x) = f * h_\theta(x) := \int_{\Omega_n} f(y) h_\theta(xy)\, dy, \qquad x \in \Omega_n.$$

Hence M_θ is a multiplier operator with the multipliers

$$\widehat{h_\theta} := \{\widehat{h_\theta}(k)\}_{k=0}^{\infty},$$

where

$$\widehat{h_\theta}(k) = \frac{1}{\int_0^\theta \sin^{n-2} t\, dt} \int_0^\theta P_k^n(\cos t) \sin^{n-2} t\, dt.$$

We see that $\widehat{h_\theta}(0) = 1$. Applying Rodrigues' formula (see [WL, p. 23, Theorem 1.2.1], or [Sz, p. 67]), we have for $k > 0$

$$\widehat{h_\theta}(k) = \frac{\sin^{n-1} \theta}{(n-1) \int_0^\theta \sin^{n-2} t\, dt} P_{k-1}^{n+2}(\cos\theta). \tag{2}$$

Hence for $f \in L(\Omega_n)$ and $m > 2$, we have

$$Y_k(T_m(\theta)(f)) = -k(k+n-2)\theta^2 (\widehat{h_\theta}(k))^m, \qquad k = 0, 1, 2, \ldots\ . \tag{3}$$

For a function $f \in L(\Omega_n)$, we denote its Cesàro means of order $\delta > -1$ by

$$\sigma_N^\delta(f) = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta Y_k(f),$$

where A_k^δ denote Cesàro numbers which are given by

$$A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(\delta+1)\Gamma(k+1)}, \qquad \delta > -1.$$

It is known that when $\delta > \frac{n-2}{2}$ and $X = L^p(\Omega_n)$ $(1 \leqslant p \leqslant \infty)$ we have

$$\sup\{||\sigma_N^\delta||_{(X,X)} : N \in \mathbb{Z}_+\} \leqslant B(\delta) < \infty,$$

where $B(\delta)$ denotes a constant depending only on δ and X (see [WL, p. 50], or [So, p. 47]). We will make use of this result.

Let $\{u_k\}_{k=0}^\infty$ be a sequence of numbers. Define $\Delta^0 u_k = u_k$, $\Delta u_k = \Delta^1 u_k = u_k - u_{k+1}$ and $\Delta^{j+1} u_k = \Delta(\Delta^j u_k)$, $j = 1, 2, \ldots$.

Lemma 1. *Suppose* $1 \leqslant p \leqslant \infty$ *and* $f \in L^p(\Omega_n)$. *Let* $\{u_k\}_{k=0}^\infty$ *be a sequence of real numbers such that*

$$\lim_{k\to\infty} u_k = 0$$

and

$$\sum_{k=0}^{\infty} |\triangle^{\ell+1} u_k| A_k^\ell = M < \infty \tag{4}$$

with $\ell \in \mathbb{N}$, $n+1 \geqslant \ell > \lambda := \frac{n-2}{2}$. *Define*

$$g(x) := \sum_{k=0}^{\infty} (\triangle^{\ell+1} u_k) A_k^\ell \sigma_k^\ell(f)(x).$$

Then

$$||g||_p \leqslant C_{n,p} M ||f||_p,$$

where $C_{n,p}$ *is a constant depending only on* n *and* p, *and*

$$Y_k(g)(x) = u_k Y_k(f)(x), \qquad k = 0, 1, \ldots .$$

Proof. Since $\sup_k ||\sigma_k^\ell(f)||_p \leqslant C_{np}||f||_p$ for $n+1 \geqslant \ell > \frac{n-2}{2}$, we know, by (4), that the series

$$\sum_{k=0}^{\infty} (\triangle^{\ell+1} u_k) A_k^\ell \sigma_k^\ell(f)(x)$$

converges absolutely in $L^p(\Omega_n)$ and $||g||_p \leqslant C_{n,p} M ||f||_p$.

Fix $k \in \mathbb{Z}_+$. Since the projection Y_k is a continuous operator from $L^p(\Omega_n)$ to $L^p(\Omega_n)$, we have

$$Y_k(g)(x) = \sum_{j=0}^{\infty} (\triangle^{\ell+1} u_j) A_j^\ell \, Y_k(\sigma_j^\ell(f))(x).$$

By the definition of Cesàro means we know that when $j < k$, $Y_k(\sigma_j^\ell(f)) = 0$ and for $j \geqslant k$

$$Y_k(\sigma_j^\ell(f)) = \frac{A_{j-k}^\ell}{A_j^\ell} \, Y_k(f).$$

Then we get

$$Y_k(g)(x) = \left(\sum_{j=k}^{\infty} \triangle^{\ell+1} u_j A_{j-k}^\ell \right) Y_k(f)(x).$$

Therefore, it is sufficient to prove

$$\sum_{j=k}^{\infty} \triangle^{\ell+1} u_j A_{j-k}^\ell = u_k. \tag{5}$$

Since $\lim_{k\to\infty} u_k = 0$, we know that for any $i \in \mathbb{N}$, $\lim_{k\to\infty} |\triangle^i u_k| = 0$ and

$$\triangle^i u_k = \sum_{j=k}^{\infty} \triangle^{i+1} u_j.$$

Consequently, noticing $\sum_{k=0}^{j} A_k^{\ell-1} = A_j^\ell$ we get

$$\sum_{k=0}^{\infty} |\triangle^\ell u_k| A_k^{\ell-1} \leqslant \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} |\triangle^{\ell+1} u_j| A_k^{\ell-1} = \sum_{j=0}^{\infty} |\triangle^{\ell+1} u_j| \sum_{k=0}^{j} A_k^{\ell-1}$$
$$= \sum_{k=0}^{\infty} |\triangle^{\ell+1} u_j| A_j^\ell < \infty.$$

Inductively, one can prove that for all $0 \leqslant i \leqslant \ell$,

$$\sum_{k=0}^{\infty} | \triangle^{i+1} u_k | A_k^i < \infty.$$

This implies

$$\lim_{k \to \infty} | \triangle^{i+1} u_k | A_k^i = 0, \qquad 0 \leqslant i \leqslant \ell.$$

For any two sequences of numbers $\{u_k\}_{k=0}^{\infty}, \{v_k\}_{k=0}^{\infty}$ we have the following Abel transformation formula:

$$\sum_{k=0}^{m+1} u_k v_k = \sum_{k=0}^{m} \Delta u_k \sum_{j=0}^{k} v_j + u_{m+1} \sum_{k=0}^{m+1} v_k.$$

If we know $\sum_{k=0}^{\infty} u_k v_k \in \mathbb{R}$ and $\lim_{m \to \infty} u_{m+1} \sum_{k=0}^{m+1} v_k = 0$, then passing to limit, we obtain

$$\sum_{k=0}^{\infty} u_k v_k = \sum_{k=0}^{\infty} \Delta u_k \sum_{j=0}^{k} v_j,$$

which will be the Abel transformation formula for our use.

Now using the Abel transformation once for our sequence $\{u_k\}_{k=0}^{\infty}$ in the lemma and a special sequence $\{v_k = r^k\}_{k=0}^{\infty}$ with $0 < r < 1$, noticing $\sum_{j=0}^{k} v_j = \sum_{k=0}^{j} A_{j-k}^0 r^k$, we get

$$\sum_{j=0}^{\infty} u_j r^j = \sum_{j=0}^{\infty} (\triangle^1 u_j) \left(\sum_{k=0}^{j} A_{j-k}^0 r^k \right).$$

Writing $v_j^1 = \sum_{k=0}^{j} A_{j-k}^0 r^k$ and applying the Abel transformation once again, we get

$$\sum_{j=0}^{\infty} u_j r^j = \sum_{j=0}^{\infty} (\triangle^2 u_j) \left(\sum_{k=0}^{j} v_k^1 \right).$$

Note that

$$\sum_{k=0}^{j} v_k^1 = \sum_{k=0}^{j} \sum_{\mu=0}^{k} A_{k-\mu}^0 r^\mu = \sum_{\mu=0}^{j} \sum_{k=\mu}^{j} A_{k-\mu}^0 r^\mu = \sum_{\mu=0}^{j} A_{j-\mu}^1 r^\mu.$$

Then we get

$$\sum_{j=0}^{\infty} u_j r^j = \sum_{j=0}^{\infty} (\triangle^2 u_j)\left(\sum_{\mu=0}^{j} A_{j-\mu}^{1} r^{\mu}\right).$$

So, using the Abel transformation inductively $\ell+1$ times, we get

$$\sum_{j=0}^{\infty} u_j r^j = \sum_{j=0}^{\infty} (\triangle^{\ell+1} u_j)\left(\sum_{k=0}^{j} A_{j-k}^{\ell} r^{k}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty} (\triangle^{\ell+1} u_j) A_{j-k}^{\ell}\right) r^k,$$
$$0<r<1.$$

Comparing the coefficients of r^k, we get (5) and complete the proof. ∎

We know (see [Sz, (4.1.1)])

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}.$$

Now we define

$$Q_k^{(\alpha,\beta)}(t) = \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\alpha,\beta)}(1)}.$$

LEMMA 2. *If $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, then there is a constant $B(\alpha,\beta)$ depending only on α and β such that for $k \geqslant 1$*

$$|Q_k^{(\alpha,\beta)}(\cos\theta)| \leqslant \begin{cases} 1, & 0 \leqslant \theta \leqslant \pi, \\ \frac{B(\alpha,\beta)}{(k\theta)^{\alpha+\frac{1}{2}}}, & 0<\theta\leqslant\frac{\pi}{2}, \\ \frac{B(\alpha,\beta)}{(k(\pi-\theta))^{\beta+\frac{1}{2}}}, & \frac{\pi}{2}\leqslant\theta\leqslant\pi, \end{cases} \tag{6}$$

and, in particular,

$$|P_k^{n+2}(\cos\theta)| \leqslant \begin{cases} 1, & 0\leqslant\theta\leqslant\pi, \\ \frac{B(n)}{(k\sin\theta)^{\frac{n}{2}}}, & \theta\in(0,\pi), \end{cases} \tag{7}$$

where $B(n)$ denotes a constant depending only on n.

Proof. By formula (7.32.2) in [Sz, p. 168]

$$\max\{|P_k^{(\alpha,\beta)}(t)| : -1 \leqslant t \leqslant 1\} = \binom{k+q}{k} \sim k^q \qquad \text{if } q = \max(\alpha, \beta) \geqslant -\frac{1}{2}$$

and formulas (4.1.1) and (4.1.4) in [Sz, pp. 58, 59]

$$P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k}, \qquad P_k^{(\alpha,\beta)}(t) = (-1)^k P_k^{(\beta,\alpha)}(-t)$$

we see that, under the condition $\alpha \geqslant \beta \geqslant \frac{n-3}{2}$, we get the first estimate in (6).

We apply formula (8.21.18) in [Sz]: for $\frac{1}{k} \leqslant \theta \leqslant \pi - \frac{1}{k}$

$$P_k^{(\alpha,\beta)}(\cos\theta) = \frac{1}{\sqrt{\pi k}(\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\beta+\frac{1}{2}}}\left(\cos\left(\left(k + \frac{\alpha+\beta+1}{2}\right)\theta - \frac{\alpha+\frac{1}{2}}{2}\pi\right) + O\left(\frac{1}{k\sin\theta}\right)\right).$$

The term $O(\frac{1}{k\sin\theta})$ can be written as $\frac{r(k,\alpha,\beta,\theta)}{k\sin\theta}$ where $|r(k,\alpha,\beta,\theta)| \leqslant B(\alpha,\beta)$ for $B(\alpha,\beta)$ being a constant depending only on α and β. Then we see that the second and third estimates in (6) are valid for $\theta \in (\frac{1}{k}, \pi - \frac{1}{k})$. Hence by the first estimate they are also valid for $0 < \theta < \frac{1}{k}$ and $\pi - \frac{1}{k} \leqslant \theta < \pi$.

Applying (6) to the case $\alpha = \beta = \frac{n-1}{2}$, we get (7). ∎

3. FURTHER LEMMAS

Define

$$u_k(m,\theta) = -k(k+n-2)\theta^2(\widehat{h_\theta}(k))^m. \tag{8}$$

For simplicity we write

$$\phi(m,\theta) = -\theta^2\left(\frac{\sin^{n-1}\theta}{(n-1)\int_0^\theta \sin^{n-2}t\,dt}\right)^m, \qquad \psi_k(m,\theta) = (P_{k-1}^{\frac{n+2}{2}}(\cos\theta))^m,$$

and hence, using (2), we have

$$u_k(m,\theta) = k(k+n-2)\phi(m,\theta)\psi_k(m,\theta) \qquad (0 < \theta < \pi). \tag{8$'$}$$

LEMMA 3. *Let $m \in \mathbb{N}$, $m > 10 + n$. Then for any $f \in L(\Omega_n)$,*

$$T_m(\theta)(f) = \sum_{k=1}^{\infty} u_k(m,\theta)\, Y_k(f) = \sum_{k=1}^{\infty} \Delta^{n+1} u_k(m,\theta) A_k^n \sigma_k^n(f).$$

Proof. By (7) we have roughly (for $\theta \in (0,\pi)$)

$$|u_k(m,\theta)| \leqslant (B(n,\theta))^m k^{-\frac{mn}{2}} \leqslant (B(n,\theta))^m k^{-5n},$$

where here and in what follows $B(n,\theta)$ denote constants depending only on n and θ which may have different values in different occurrences. Of course, for $j = 1, \ldots, n+1$ the estimates

$$|\Delta^j u_k(m,\theta)| \leqslant (B(n,\theta))^m k^{-\frac{mn}{2}} \leqslant (B(n,\theta))^m k^{-5n}$$

hold. So, the conditions of Lemma 1 are satisfied and hence Lemma 3 is valid. ∎

LEMMA 4. *When $m > 10$*

$$\sup\left\{ ||T_m(\theta)||_{(X,X)} : \theta \in \left[\frac{\pi}{2}, \pi\right) \right\} \leqslant C(n) \pi^{-\frac{m}{2}}.$$

Proof. Throughout this paper we use $C(n)$ to denote constants depending only on n which may have different values in different occurrences.

By Lemma 3 we have

$$||T_m(\theta)||_{(X,X)} \leqslant C(n) \sum_{k=1}^{\infty} k^n |\Delta^{n+1} u_k(m,\theta)|.$$

Since $\theta \in [\frac{\pi}{2}, \pi)$, we have

$$(n-1) \int_0^{\theta} \sin^{n-2} t\, dt \geqslant (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} t\, dt = \frac{\Gamma(\frac{n+1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} > \sqrt{\pi}.$$

Write $\xi = \pi - \theta$. Then we get

$$|\phi(m,\theta)| \leqslant \pi^{-\frac{m}{2}+2} (\sin \xi)^{(n-1)m}. \tag{9}$$

We fix the constant $B(n)$ in (7) of Lemma 2 as $b = B(n) > 1$. Define

$$I_1 = \left\{ k \in \mathbb{N} : k \leqslant \frac{b}{\sin(\pi - \theta)} \right\}, \qquad I_2 = \left\{ k \in \mathbb{N} : k > \frac{b}{\sin(\pi - \theta)} \right\}.$$

When $k \in I_1$ we use the estimate $|\psi_k(m,\theta)| \leqslant 1$ $(\forall k \in \mathbb{N})$ and hence by (8′)

$$|u_k(m,\theta)| \leqslant (n-1)k^2(\sin\xi)^{(n-1)m}\pi^{-\frac{m}{2}-2}. \tag{10}$$

Since

$$\Delta^{n+1}u_k(m,\theta) = \sum_{v=0}^{n+1} C_{n+1}^{v}(-1)^v u_{k+v}(m,\theta),$$

we get, by (10)

$$|\Delta^{n+1}u_k(m,\theta)| \leqslant C(n)k^2(\sin\xi)^{(n-1)m}\pi^{-\frac{m}{2}}. \tag{11}$$

Therefore, we get (for $m > 10$)

$$\sum_{k\in I_1} k^n|\Delta^{n+1}u_k(m,\theta)| \leqslant C(n)\xi^{(n-1)m}\pi^{-\frac{m}{2}}\xi^{-n-3} \leqslant C(n)\pi^{-\frac{m}{2}}. \tag{12}$$

When $k \in I_2$, i.e. $k > \frac{b}{\sin\xi}$, we apply (7) of Lemma 2. Then we get

$$|\psi_k(m,\theta)| \leqslant \frac{b^m}{(k\sin\xi)^{\frac{mn}{2}}}$$

and hence by (9)

$$|u_k(m,\theta)| \leqslant (\sin\xi)^{(n-1)m}\pi^{-\frac{m}{2}+2}\frac{(n-1)k^2b^m}{(k\sin\xi)^{\frac{mn}{2}}}. \tag{13}$$

Therefore

$$|\Delta^{n+1}u_k(m,\theta)| \leqslant C(n)(\sin\xi)^{(n-1)m}\pi^{-\frac{m}{2}}\frac{k^2b^m}{(k\sin\xi)^{\frac{mn}{2}}}. \tag{14}$$

So we get

$$\sum_{k\in I_2} k^n|\Delta^{n+1}u_k(m,\theta)| \leqslant C(n)\pi^{-\frac{m}{2}}\sum_{k\in I_2} k^{n+2-\frac{mn}{2}}b^m \leqslant C(n)\pi^{-\frac{m}{2}}. \tag{15}$$

A combination of (12) and (15) yields the conclusion of the lemma. ∎

4. THE CASE $0 < \theta < \frac{\pi}{2}$

In what follows we assume $\theta \in (0, \frac{\pi}{2})$. We may assume $m > 10 + n$.

In order to find a suitable estimate for $|\Delta^{n+1}u_k(m,\theta)|$ we first establish the following lemma.

LEMMA 5. *Let* $m \in \mathbb{N}$, $m > 10+n$. *Assume* $\{h_k(\mu)\}_{k=1}^{\infty}$, $\mu = 1, \ldots, m$, *are sequences of numbers. Then*

$$\Delta^j\left(\prod_{\mu=1}^{m} h_k(\mu)\right) = \sum_{\ell_1=1}^{m}\cdots\sum_{\ell_j=1}^{m}\left(\prod_{\mu=1}^{m} h_k(\ell_1,\ldots,\ell_j,\mu)\right), \qquad j \in \mathbb{N}, \quad (16)$$

where the numbers $h_k(\ell_1,\ldots,\ell_j,\mu)$ *are defined inductively as follows*:

$$h_k(\ell_1,\mu) = \begin{cases} h_k(\mu) & \text{when } 1 \leqslant \mu < \ell_1, \\ \Delta h_k(\mu) & \text{when } \mu = \ell_1, \\ h_{k+1}(\mu) & \text{when } \ell_1 < \mu \leqslant m, \end{cases} \quad (17)$$

$$h_k(\ell_1,\ldots,\ell_{j+1},\mu) = \begin{cases} h_k(\ell_1,\ldots,\ell_j,\mu) & \text{when } 1 \leqslant \mu < \ell_{j+1}, \\ \Delta h_k(\ell_1,\ldots,\ell_j,\mu) & \text{when } \mu = \ell_{j+1}, \\ h_{k+1}(\ell_1,\ldots,\ell_j,\mu) & \text{when } \ell_{j+1} < \mu \leqslant m. \end{cases} \quad (18)$$

Proof. It is easy to verify (16) for $j = 1$, i.e.

$$\Delta\left(\prod_{\mu=1}^{m} h_k(\mu)\right) = \sum_{\ell=1}^{m}\left(\prod_{\mu=1}^{m} h_k(\ell,\mu)\right),$$

where $h_k(\ell,\mu)$ is defined as in (17). Suppose (16) is valid for j. Then we have

$$\Delta^{j+1}\left(\prod_{\mu=1}^{m} h_k(\mu)\right) = \sum_{\ell_1=1}^{m}\cdots\sum_{\ell_j=1}^{m}\Delta\left(\prod_{\mu=1}^{m} h_k(\ell_1,\ldots,\ell_j,\mu)\right).$$

But by the result for $j = 1$ we have

$$\Delta\left(\prod_{\mu=1}^{m} h_k(\ell_1,\ldots,\ell_j,\mu)\right) = \sum_{\ell_{j+1}=1}^{m}\prod_{\mu=1}^{m} h_k(\ell_1,\ldots,\ell_j,\ell_{j+1},\mu)),$$

where $h_k(\ell_1, \ldots, \ell_j, \ell_{j+1}, \mu)$ is defined in (18). Then we have proved that (16) is valid for all $j \in \mathbb{N}$ and finish the proof. ∎

Let us now estimate $|\Delta^j P_{k-1}^{n+2}(\cos\theta)|,\ j = 1, \ldots, n+1$. We apply formula (4.5.4) in [Sz, p. 71]:

$$P_k^{(\alpha+1,\beta)}(x) = \frac{2}{2k+\alpha+\beta+2} \frac{(k+\alpha+1)P_k^{(\alpha,\beta)}(x) - (k+1)P_{k+1}^{(\alpha,\beta)}(x)}{1-x}$$

and get

$$Q_k^{(\alpha,\beta)}(x) - Q_{k+1}^{(\alpha,\beta)}(x) = (1-x)\frac{2k+\alpha+\beta+2}{2(\alpha+1)} Q_k^{(\alpha+1,\beta)}(x). \tag{19}$$

In particular, with $\alpha = \beta = \frac{n-1}{2}$, we have

$$P_k^{n+2}(\cos\theta) - P_{k+1}^{n+2}(\cos\theta) = \frac{2k+n+1}{n+1}(1-\cos\theta) Q_k^{(\frac{n+1}{2},\frac{n-1}{2})}(\cos\theta). \tag{20}$$

Applying (19), (20) inductively and making use of Lemma 2, we get

$$|\Delta^j P_{k-1}^{n+2}(\cos\theta)| \leqslant \begin{cases} \frac{B\theta^j}{(k\theta)^{\frac{n}{2}}} & \text{when } k\theta \geqslant 1, \\ B\theta^j & \text{when } k\theta < 1, \end{cases} \qquad j = 0, 1, \ldots, n+1, \tag{21}$$

where $B > 1$ denotes a constant depending only on n.

Using the constant B in (21), we will treat the cases $k\theta > 2B$ and $k\theta \leqslant 2B$ separately.

Lemma 6. *Let $m \in \mathbb{N},\ m > 10n$ and $0 < \theta < \frac{\pi}{2}$. If $0 < \theta < \frac{\pi}{2}$ and $k\theta \geqslant 2B$ with B the constant in* (21), *then*

$$|\Delta^{n+1} u_k(m,\theta)| \leqslant C(n) m^{n+1} \theta^{n+1} \left(\frac{B}{(k\theta)^{\frac{n}{2}}}\right)^{m-n}, \tag{22}$$

where $C(n)$ denotes a constant depending only on n.

Proof. Recall (see the proof of Lemma 4)

$$u_k(m,\theta) = \phi(m,\theta)(k(k+n-2)\psi_k(m,\theta).$$

For any sequence $u_k = a_k b_k$ we can easily verify by induction that

$$\Delta^{n+1} u_k = \sum_{j=0}^{n+1} C_{n+1}^{j} \Delta^{j} a_{k+n+1-j} \Delta^{n+1-j} b_k. \tag{23}$$

Applying (23) with $a_k = k(k+n-2),\ b_k = \psi_k(m,\theta)$, we get

$$\begin{aligned}
&\Delta^{n+1} u_k(m,\theta)\\
&\quad = \phi(m,\theta)\big(\Delta^{n+1}\psi_k(m,\theta) a_{k+n+1} + (n+1)\Delta^{n}\psi_k(m,\theta)\Delta a_{k+n}\\
&\qquad + \frac{n(n+1)}{2}\Delta^{n-1}\psi_k(m,\theta)\Delta^2 a_{k+n-1}\Big)\\
&\quad = \phi(m,\theta)((k+n+1)(k+2n-1)\Delta^{n+1}\psi_k(m,\theta) + (n+1)(-2k-3n+1)\\
&\qquad \times \Delta^{n}\psi_k(m,\theta) + n(n+1)\Delta^{n-1}\psi_k(m,\theta)).
\end{aligned}$$

Hence

$$\begin{aligned}
|\Delta^{n+1} u_k(m,\theta)| \leqslant 3(n+1)^2 |\phi(m,\theta)| (k^2 |\Delta^{n+1}\psi_k(m,\theta)| + k|\Delta^{n}\psi_k(m,\theta)|\\
+ |\Delta^{n-1}\psi_k(m,\theta)|).
\end{aligned} \tag{24}$$

We apply Lemma 5 by writing $h_k = h_k(\mu) = P_{k-1}^{n+2}(\cos\theta),\ \mu = 1, \ldots, m$. Then by (16) we have

$$\Delta^{j}\psi_k(m,\theta) = \sum_{\ell_1=1}^{m} \cdots \sum_{\ell_j=1}^{m} \left(\prod_{\mu=1}^{m} h_k(\ell_1, \ldots, \ell_j, \mu) \right) \qquad (j = n-1, n, n+1). \tag{25}$$

In our case, by (17) we have

$$h_k(\ell_1,\mu) = \begin{cases} h_k(\mu) = h_k & \text{when } 1 \leqslant \mu < \ell_1, \\ \Delta h_k(\mu) = \Delta h_k & \text{when } \mu = \ell_1, \\ h_{k+1}(\mu) = h_{k+1} & \text{when } \ell_1 < \mu \leqslant m, \end{cases} \tag{26}$$

and then by (18)

$$h_k(\ell_1,\ell_2,\mu) = \begin{cases} h_k(\ell_1,\mu) = h_k & \text{when } 1 \leqslant \mu < \min(\ell_1,\ell_2), \\ \Delta h_k(\ell_1,\mu) = \Delta h_k & \text{when } \mu = \ell_1 < \ell_2, \\ \Delta h_k(\ell_1,\mu) = h_{k+1} & \text{when } \ell_1 < \mu < \ell_2, \\ \Delta h_k(\ell_1,\mu) = \Delta h_k & \text{when } \ell_2 = \mu < \ell_1, \\ \Delta h_k(\ell_1,\mu) = \Delta^2 h_k & \text{when } \mu = \ell_1 = \ell_2, \\ \Delta h_k(\ell_1,\mu) = \Delta h_{k+1} & \text{when } \ell_1 < \mu = \ell_2, \\ h_{k+1}(\ell_1,\mu) = h_{k+1} & \text{when } \ell_2 < \mu < \ell_1 \\ h_{k+1}(\ell_1,\mu) = \Delta h_{k+1} & \text{when } \ell_2 < \mu = \ell_1 \\ h_{k+1}(\ell_1,\mu) = h_{k+2} & \text{when } \ell_1 < \ell_2 < \mu. \end{cases} \tag{27}$$

From (26) and (27) by using induction we conclude that for all $j \in \mathbb{N}$

$$h_k(\ell_1,\ldots,\ell_j,\mu) \in \{h_s, \Delta^t h_{k+j-t} : s = k, k+1, \ldots, k+j,\ \ t = 1,\ldots,j\}.$$

Furthermore, by induction we see that in each product

$$\prod_{\mu=1}^{m} h_k(\ell_1,\ldots,\ell_j,\mu)$$

the factors having form "h_s" appear totally $m-j$ times and the sum of all degrees "t" over all factors having the form $\Delta^t h_{k+j-t}$ is exactly j. In effect, for any $(\ell_1,\ldots,\ell_j)$ $(j \leqslant n+1)$ we define

$$I_1(\ell_1,\ldots,\ell_j) = \{\mu : h_k(\ell_1,\ldots,\ell_j,\mu) \in \{h_k, h_{k+1},\ldots,h_{k+j}\}\},$$

$$I_2(\ell_1,\ldots,\ell_j) = \{\mu : h_k(\ell_1,\ldots,\ell_j,\mu) \in \{\Delta^t h_{k+j-t} : t = 1,\ldots,j\}\}.$$

Then the cardinality of I_1 is exactly $m-j$. Hence by (21) we have

$$\left|\prod_{\mu\in I_1} h_k(\ell_1,\ldots,\ell_j,\mu)\right| \leqslant \left(\frac{B}{(k\theta)^{\frac{n}{2}}}\right)^{m-j}.$$

Meanwhile, by (21)

$$\left|\prod_{\mu\in I_2} h_k(\ell_1,\ldots,\ell_j,\mu)\right| \leqslant \frac{(B\theta)^j}{(k\theta)^{\frac{n}{2}}}.$$

Then we get

$$\left|\prod_{\mu=1}^{m} h_k(\ell_1,\ldots,\ell_j,\mu)\right| \leqslant \frac{B^m\theta^j}{(k\theta)^{(m-j+1)\frac{n}{2}}}.$$

Then by (25) we get

$$|\Delta^j\psi_k(m,\theta)| \leqslant \frac{B^m m^j\theta^j}{(k\theta)^{(m-j+1)\frac{n}{2}}}, \qquad j=n-1,n,n+1 \quad (k\theta \geqslant 2B). \tag{28}$$

Substituting (28) into (24) and observing $|\phi(m,\theta)| \leqslant \theta^2$, we obtain

$$|\Delta^{n+1}u_k(m,\theta)| \leqslant 9(n+1)^2 \frac{B^m m^{n+1}\theta^{n+1}}{(k\theta)^{\frac{(m-n)n}{2}}},$$

and complete the proof. ∎

Now we consider the case $0<k\theta \leqslant 2(n+1)B$.

LEMMA 7. *Let $m\in\mathbb{N}$, $m>10n$ and $0<k\theta\leqslant 2(n+1)B$, then*

$$|\widehat{h_\theta}(k)| \leqslant \begin{cases} 1-\frac{1}{10(n+2)}(k\theta)^2 & \text{when } 0\leqslant k\theta\leqslant\frac{1}{2}, \\ \delta(n) & \text{when } \frac{1}{2(n+1)}\leqslant k\theta\leqslant 2(n+1)B, \end{cases}$$

where $\delta(n) = 1-\frac{1}{(2(n+1)^2\pi B)^{n+2}}$.

Proof. From the formula

$$\frac{d}{dt}P_k^n(t) = \frac{k(k+n-2)}{n-1}P_{k-1}^{n+2}(t)$$

(see [WL, p. 31, Corollary 1.2.8] or [Sz, p. 81, (4.7.14)]) and the Lagrange mean value theorem we know that there is a value $\xi\in(0,\theta)$ such that

$$1-P_k^n(\cos\theta) = \frac{k(k+n-2)}{n-1}P_{k-1}^{n+2}(\cos\xi)(1-\cos\theta). \tag{29}$$

Hence

$$0 \leqslant 1 - P_k^n(\cos\theta) \leqslant (k\theta)^2.$$

So, if $0 < k\theta \leqslant \frac{1}{2}$, then

$$P_k^n(\cos\theta) \geqslant \frac{3}{4}. \tag{30}$$

Of course (30) is also valid for $P_{k-1}^{n+2}(\cos\theta)$. From (29) applying (30) to $P_{k-1}^{n+2}(\cos\xi)$ we derive

$$\frac{3}{4} \leqslant P_k^n(\cos\theta) \leqslant 1 - \frac{1}{10n}(k\theta)^2 \qquad \text{when } 0 \leqslant k\theta \leqslant \frac{1}{2}.$$

Then we get

$$\frac{3}{4} \leqslant P_{k-1}^{n+2}(\cos\theta) \leqslant 1 - \frac{1}{10(n+2)}(k\theta)^2 \qquad \text{when } 0 \leqslant k\theta \leqslant \frac{1}{2}. \tag{31}$$

By (2) and (31) we have

$$|\widehat{h_\theta}(k)| \leqslant 1 - \frac{1}{10(n+2)}(k\theta)^2 \qquad \text{when } 0 \leqslant k\theta \leqslant \frac{1}{2}. \tag{32}$$

When $\frac{1}{2(n+1)k} \leqslant \theta \leqslant 2(n+1)B\frac{1}{k}$, i.e. $\frac{1}{2(n+1)} \leqslant k\theta \leqslant 2(n+1)B$, we have

$$\begin{aligned}
\widehat{h_\theta}(k) &= 1 - \frac{1}{\int_0^\theta \sin^{n-2} t\,dt}\int_0^\theta (1 - P_k^n(\cos t))\sin^{n-2} t\,dt \\
&\leqslant 1 - \frac{1}{\int_0^\theta \sin^{n-2} t\,dt}\int_0^{\frac{1}{2(n+1)k}} (1 - P_k^n(\cos t))\sin^{n-2} t\,dt \\
&= 1 - \frac{1}{\int_0^\theta \sin^{n-2} t\,dt}\int_0^{\frac{1}{2(n+1)k}} \frac{k(k+n-2)}{n-2} \\
&\quad \times \left(\int_0^t P_{k-1}^{n+2}(\cos u)\sin u\,du\right)\sin^{n-2} t\,dt.
\end{aligned}$$

Applying (30) we get

$$\begin{aligned}
&\frac{1}{\int_0^\theta \sin^{n-2} t\,dt}\int_0^{\frac{1}{2(n+1)k}} \frac{k(k+n-2)}{n-2}\left(\int_0^t P_{k-1}^{n+2}(\cos u)\sin u\,du\right)\sin^{n-2} t\,dt \\
&\geqslant \frac{n-1}{\theta^{n-1}}\int_0^{\frac{1}{2(n+1)k}} \frac{k(k+n-2)}{n-2}\left(\int_0^t \frac{3}{4}\sin u\,du\right)\sin^{n-2} t\,dt
\end{aligned}$$

$$\geqslant \frac{k(k+n-2)}{n-2}\frac{3}{4}\frac{n-1}{\theta^{n-1}}\int_0^{\frac{1}{2(n+1)k}} 2\sin^2\frac{t}{2}\sin^{n-2}t\,dt$$

$$\geqslant \frac{3(n-1)k(k+n-2)}{8(n-2)\theta^{n-1}}\left(\frac{2}{\pi}\right)^n\int_0^{\frac{1}{2(n+1)k}} t^n\,dt \geqslant \frac{1}{(2(n+1)^2\pi B)^{n+2}}.$$

Therefore, when $\frac{1}{2(n+1)} \leqslant k\theta \leqslant 2(n+1)B$,

$$\widehat{h_\theta}(k) \leqslant 1 - \frac{1}{(2(n+1)^2\pi B)^{n+2}}.$$

Thus

$$-\widehat{h_\theta}(k) = 1 - \frac{1}{\int_0^\theta \sin^{n-2}t\,dt}\int_0^\theta (1+P_k^n(\cos t))\sin^{n-2}t\,dt$$

$$\leqslant 1 - \frac{1}{\int_0^\theta t^{n-2}\,dt}\int_0^{\frac{1}{2k}} \frac{7}{4}\left(\frac{2}{\pi}\right)^{n-2} t^{n-2}\,dt = 1 - \frac{7}{8\pi^{n-2}(k\theta)^{n-1}}$$

$$\leqslant 1 - \frac{7\pi^2 B}{8(\pi B)^n} < 1 - \frac{1}{(2(n+1)^2\pi B)^{n+2}}.$$

Then we get for $\delta(n) = 1 - \frac{1}{(2(n+1)^2\pi B)^{n+2}}$

$$|\widehat{h_\theta}(k)| \leqslant \delta(n) < 1, \qquad \text{when } \frac{1}{2(n+1)} \leqslant k\theta \leqslant 2(n+1)B. \tag{33}$$

A combination of (32) and (33) completes the proof. ∎

LEMMA 8. *Let $m \in \mathbb{N}$, $m > 10n$ and let B be the constant in* (21). *Then*

$$|\Delta^{n+1}u_k(m,\theta)| \leqslant \begin{cases} C(n)m^{n+1}\theta^{n+1}\left(1 - \frac{1}{10(n+2)}(k\theta)^2\right)^m & \text{when } 0 \leqslant k\theta \leqslant \frac{1}{2(n+1)}, \\ C(n)m^{n+1}\theta^{n+1}(\delta(n))^m & \text{when } \frac{1}{2(n+1)} \leqslant k\theta \leqslant 2B, \end{cases}$$

where $\delta(n) = 1 - \frac{1}{(2(n+1)^2\pi B)^{n+2}}$.

Proof. Applying (23) with $a_k = k(k+n-2)$, $b_k = (\widehat{h_\theta}(k))^m$ we get

$$\Delta^{n+1}u_k(m,\theta)$$

$$= -\theta^2\left(\Delta^{n+1}b_k a_{k+n+1} + (n+1)\Delta^n b_k \Delta a_{k+n} + \frac{n(n+1)}{2}\Delta^{n-1}b_k\Delta^2 a_{k+n-1}\right)$$

$$= -\theta^2((k+n+1)(k+2n-1)\Delta^{n+1}(\widehat{h_\theta}(k))^m$$

$$+ (n+1)(-2k-3n+1)\Delta^n(\widehat{h_\theta}(k))^m + n(n+1)\Delta^{n-1}(\widehat{h_\theta}(k))^m).$$

Hence

$$|\Delta^{n+1}u_k(m,\theta)| \leqslant 3(n+1)^2\theta^2(k^2|\Delta^{n+1}(\widehat{h}_\theta(k))^m| + k|\Delta^n(\widehat{h}_\theta(k))^m| + |\Delta^{n-1}(\widehat{h}_\theta(k))^m|). \tag{34}$$

Now we repeat the same argument as in the proof of Lemma 6. This time we apply Lemma 5 to $\widehat{h}_\theta(k)$. We take $\widehat{h}_\theta(k)$ in the place of $h_k(\mu) = \widehat{h}_\theta(k),\ \mu = 1,\dots,m$ in Lemma 5. Then by (16) we have

$$\Delta^j(\widehat{h}_\theta(k))^m = \sum_{\ell_1=1}^{m}\cdots\sum_{\ell_j=1}^{m}\left(\prod_{\mu=1}^{m} h_k(\ell_1,\dots,\ell_j,\mu)\right) \qquad (j = n-1, n, n+1), \tag{35}$$

where, by (17),

$$h_k(\ell_1,\mu) = \begin{cases} \widehat{h}_\theta(k) & \text{when } 1 \leqslant \mu < \ell_1, \\ \Delta(\widehat{h}_\theta(k)) & \text{when } \mu = \ell_1, \\ \widehat{h}_\theta(k+1) & \text{when } \ell_1 < \mu \leqslant m, \end{cases} \tag{36}$$

and then by (18)

$$h_k(\ell_1,\ell_2,\mu) = \begin{cases} h_k(\ell_1,\mu) = \widehat{h}_\theta(k) & \text{when } 1 \leqslant \mu < \min(\ell_1,\ell_2), \\ \Delta h_k(\ell_1,\mu) = \Delta\widehat{h}_\theta(k) & \text{when } \mu = \ell_1 < \ell_2, \\ \Delta h_k(\ell_1,\mu) = \widehat{h}_\theta(k+1) & \text{when } \ell_1 < \mu < \ell_2, \\ \Delta h_k(\ell_1,\mu) = \Delta\widehat{h}_\theta(k) & \text{when } \ell_2 = \mu < \ell_1, \\ \Delta h_k(\ell_1,\mu) = \Delta^2\widehat{h}_\theta(k) & \text{when } \mu = \ell_1 = \ell_2, \\ \Delta h_k(\ell_1,\mu) = \Delta\widehat{h}_\theta(k+1) & \text{when } \ell_1 < \mu = \ell_2, \\ h_{k+1}(\ell_1,\mu) = \widehat{h}_\theta(k+1) & \text{when } \ell_2 < \mu < \ell_1 \\ h_{k+1}(\ell_1,\mu) = \Delta\widehat{h}_\theta(k+1) & \text{when } \ell_2 < \mu = \ell_1 \\ h_{k+1}(\ell_1,\mu) = \widehat{h}_\theta(k+2) & \text{when } \ell_1 < \ell_2 < \mu. \end{cases} \tag{37}$$

From (36) and (37) by using induction, we conclude that for all $j \in \mathbb{N}$

$$h_k(\ell_1,\dots,\ell_j,\mu) \in \{\widehat{h}_\theta(s), \Delta^t\widehat{h}_\theta(k+j-t) : s = k, k+1, \dots, k+j,\ t = 1,\dots,j\}.$$

Furthermore, by induction we see that in each product

$$\prod_{\mu=1}^{m} h_k(\ell_1,\dots,\ell_j,\mu)$$

the factors having form "$\widehat{h_\theta}(s)$" appear totally $m-j$ times and the sum of all degrees "t" over all factors having the form $\Delta^t\widehat{h_\theta}(k+j-t)$ is exactly j. In effect, for any $(\ell_1,\ldots,\ell_j)$ $(j\leqslant n+1)$ we define

$$I_1(\ell_1,\ldots,\ell_j) = \{\mu : h_k(\ell_1,\ldots,\ell_j,\mu) \in \{\widehat{h_\theta}(k),\widehat{h_\theta}(k+1),\ldots,\widehat{h_\theta}(k+j)\}\},$$

$$I_2(\ell_1,\ldots,\ell_j) = \{\mu : h_k(\ell_1,\ldots,\ell_j,\mu) \in \{\Delta^t\widehat{h_\theta}(k+j-t) : t=1,\ldots,j\}\}.$$

Then the cardinality of I_1 is exactly $m-j$. Hence by Lemma 7 we get

$$\left|\prod_{\mu\in I_1} h_k(\ell_1,\ldots,\ell_j,\mu)\right| \leqslant \begin{cases} (1-\frac{1}{10(n+2)}(k\theta)^2)^{m-j} & \text{when } 0\leqslant k\theta\leqslant\frac{1}{2(n+1)}, \\ (\delta(n))^{m-j} & \text{when } \frac{1}{2(n+1)}\leqslant k\theta\leqslant 2B. \end{cases} \tag{38}$$

On the other hand, we write

$$h(\theta) = \frac{\sin^{n-1}\theta}{(n-1)\int_0^\theta \sin^{n-2} t\, dt}.$$

It is obvious that $|h(\theta)|\leqslant 1$. By (2) we see

$$\widehat{h_\theta}(k) = h(\theta)P_{k-1}^{n+2}(\cos\theta).$$

Hence

$$\Delta^t h_s = h(\theta)\Delta^t P_{s-1}^{n+2}(\cos\theta), \qquad s\in\{k,k+1,\ldots,k+n\}.$$

Then by (21) we have

$$\left|\prod_{\mu\in I_2} h_k(\ell_1,\ldots,\ell_j,\mu)\right| \leqslant (B\theta)^j. \tag{39}$$

Combining (38) and (39) we get

$$\left|\prod_{\mu=1}^{m} h_k(\ell_1,\ldots,\ell_j,\mu)\right| \leqslant \begin{cases} (1-\frac{1}{10(n+2)}(k\theta)^2)^{m-j}(B\theta)^j & \text{when } 0\leqslant k\theta\leqslant\frac{1}{2(n+1)}, \\ (\delta(n))^{m-j}(B\theta)^j & \text{when } \frac{1}{2(n+1)}\leqslant k\theta\leqslant 2B. \end{cases}$$

Then by (35) we get

$$|\Delta^j(\widehat{h_\theta}(k))^m| \leqslant \begin{cases} m^{n+1}(1-\frac{1}{10(n+2)}(k\theta)^2)^{m-j}(B\theta)^j & \text{when } 0\leqslant k\theta\leqslant\frac{1}{2(n+1)}, \\ m^{n+1}(\delta(n))^{m-j}(B\theta)^j & \text{when } \frac{1}{2(n+1)}\leqslant k\theta\leqslant 2B. \end{cases} \tag{40}$$

Substituting (40) into (34), we complete the proof. ∎

5. PROOF OF THE THEOREM

For the constant B in (21), $m > 10n$ and $\theta \in (0, \frac{\pi}{2})$, we define

$$J_1 = J_1(m,\theta) = \{k \in \mathbb{N} : k\theta \leqslant m^{-\frac{1}{4}}\},$$

$$J_2 = J_2(m,\theta) = \left\{k \in \mathbb{N} : m^{-\frac{1}{4}} < k\theta \leqslant \frac{1}{2(n+1)}\right\},$$

$$J_3 = J_3(m,\theta) = \left\{k \in \mathbb{N} : \frac{1}{2(n+1)} < k\theta < 2B\right\},$$

$$J_4 = J_4(m,\theta) = \{k \in \mathbb{N} : k\theta \geqslant 2B\}.$$

Assume $J_1 \neq \emptyset$.

Take a function $\eta \in C^\infty[0,\infty)$ such that $\chi_{[0,1]} \leqslant \eta \leqslant \chi_{[0,2]}$. Write $N = \left[\frac{1}{\theta m^{\frac{1}{4}}}\right]$ and define

$$\eta_N(f) = \sum_{k=0}^{\infty} \eta\left(\frac{k}{N}\right) Y_k(f), \qquad f \in L(\Omega_n).$$

Write $|| \ ||$ instead of $|| \ ||_X$ (or $|| \ ||_{(X,X)}$) for simplicity. We have (see [WL, Theorem 4.6.3, pp. 191, 192], or [R])

$$||D(\eta_N f)|| \leqslant C(n)N^2||f||,$$

where here and in what follows we use $C(n)$ to denote constants depending only on n and the choice of η which may have different values in different occurrences. Hence for any $f \in X$

$$||T_m(\theta)(\eta_N f)|| = \theta^2||(M_\theta)^m(D(\eta_N f)|| \leqslant \frac{C(n)}{\sqrt{m}}||f||. \tag{41}$$

Now we have for any $f \in X$

$$T_m(\theta)(f) = T_m(\theta)(f - \eta_N(f)) + T_m(\theta)(\eta_N f).$$

Then

$$||T_m(\theta)(f)|| \leqslant ||T_m(\theta)(f - \eta_N(f))|| + \frac{C(n)}{\sqrt{m}}||f||. \tag{42}$$

Write $g = f - \eta_N(f)$. We know $||g|| \leqslant C||f||$ with constant C depending only on the choice of η (see [WL, p. 162], or [R]). Applying Lemma 1, we get

$$T_m(\theta)(g) = \sum_{k=0}^{\infty} \Delta^{n+1} u_k(m,\theta) A_k^n \sigma_k^n(g).$$

Note that $Y_k(g) = Y_k(f) - \eta(\frac{k}{N}) Y_k(f) = 0$ when $k \leqslant N$. We get

$$T_m(\theta)(g) = \left(\sum_{k\in J_2} + \sum_{k\in J_3} + \sum_{k\in J_4}\right) \Delta^{n+1} u_k(m,\theta) A_k^n \sigma_k^n(g). \tag{43}$$

By Lemma 6 we have for $m > 10n$

$$\begin{aligned} \left\| \sum_{k\in J_4} \Delta^{n+1} u_k(m,\theta) A_k^n \sigma_k^n(g) \right\| &\leqslant C(n) m^{n+1} ||g|| \sum_{k\in J_4} k^n \theta^{n+1} \left(\frac{B}{(k\theta)^{\frac{n}{2}}} \right)^{m-n} \\ &\leqslant C(n) \frac{m^{n+1}}{(2B)^m} ||f||. \end{aligned} \tag{44}$$

From (42)–(44) we derive

$$\begin{aligned} ||T_m(\theta)|| \leqslant C(n) &\left(\sum_{k\in(J_2 \bigcup J_3)} |\Delta^{n+1} u_k(m,\theta)| k^n + \left(\frac{1}{\sqrt{m}} + \frac{m^{n+1}}{(2B)^m} \right) \right), \\ &0 < \theta < \frac{\pi}{2}. \end{aligned} \tag{45}$$

Now we apply Lemma 8 for $k \in (J_2 \bigcup J_3)$. By Lemma 8, when $k \in J_2$

$$\begin{aligned} |\Delta^{n+1} u_k(m,\theta)| &\leqslant C(n) m^{n+1} \theta^{n+1} \left(1 - \frac{1}{10(n+2)} (k\theta)^2 \right)^m \\ &\leqslant C(n) m^{n+1} \theta^{n+1} \left(1 - \frac{1}{10(n+2)\sqrt{m}} \right)^m. \end{aligned}$$

Define

$$\gamma(n) = \sup \left\{ \left(1 - \frac{1}{10(n+2)\sqrt{m}} \right)^{\sqrt{m}} : m \geqslant n \right\}.$$

Then we see $0<\gamma(n)<1$. Hence

$$\sum_{k\in J_2} |\Delta^{n+1}u_k(m,\theta)|k^n \leqslant C(n)m^{n+1}(\gamma(n))^{\sqrt{m}}. \tag{46}$$

When $k \in J_3$, by Lemma 8,

$$|\Delta^{n+1}u_k(m,\theta)| \leqslant C(n)m^{n+1}\theta^{n+1}(\delta(n))^m,$$

where $0<\delta(n) = 1 - \frac{1}{(2(n+1)^2\pi B)^{n+2}} < 1$. Then we get

$$\sum_{k\in J_3} |\Delta^{n+1}u_k(m,\theta)|k^n \leqslant C(n)m^{n+1}(\delta(n))^m. \tag{47}$$

Substituting (46) and (47) into (45), we obtain

$$\begin{aligned} ||T_m(\theta)|| &\leqslant C(n)\left(m^{n+1}(\gamma(n))^{\sqrt{m}} + m^{n+1}(\delta(n))^m + \frac{1}{\sqrt{m}} + \frac{m^{n+1}}{(2B)^m}\right), \\ &0<\theta<\frac{\pi}{2}. \end{aligned} \tag{48}$$

Lemma 4 tells that

$$\sup\left\{||T_m(\theta)||_{(X,X)} : \theta \in \left[\frac{\pi}{2}, \pi\right)\right\} \leqslant C(n)\pi^{-\frac{m}{2}}. \tag{49}$$

Combining (48) and (49), we obtain (for $m > 10n$)

$$\begin{aligned} &\sup\{||T_m(\theta)||_{(X,X)} : \theta \in (0,\pi)\} \\ &\leqslant C(n)\left(m^{n+1}(\gamma(n))^{\sqrt{m}} + m^{n+1}(\delta(n))^m + \frac{1}{\sqrt{m}} + \frac{m^{n+1}}{(2B)^m} + \pi^{-\frac{m}{2}}\right), \end{aligned} \tag{50}$$

which completes the proof of the Theorem. ∎

Remark. The theorem can be extended by the same argument without any difficulty to the case when taking the fractional derivatives of the Laplace–Beltrami operator instead of the Laplace–Beltrami operator itself. Precisely, let $r > 0$ and $D^{\frac{r}{2}}$ denote the derivative operator of degree r which is defined (see [WL, p. 171, Definition 4.3.1]) by

$$D^{\frac{r}{2}}(f) = \sum_{k=1}^{\infty} e^{ir\frac{\pi}{2}}(k(k+n-2))^{\frac{r}{2}}Y_k(f),$$

where $i^2 = -1$. Then

$$\lim_{m\to\infty} \sup\{||\theta^r D^{\frac{r}{2}}(M_\theta)^m||_{(X,X)} : 0<\theta<\pi\} = 0.$$

For this extension we have only to replace $k(k+n-2)\theta^2$ by $(k(k+n-2)\theta^2)^{\frac{r}{2}}$.

In [D] a different definition is given for which the result follows from the results for powers of the Laplace–Beltrami operators and a Kolmogorov-type inequality in [D].

ACKNOWLEDGMENTS

The authors thank the referee very much for his valuable suggestions and pointing out misprints.

REFERENCES

[D] Z. Ditzian, Fractional derivatives and best approximation, *Acta Math. Hungar.* **81** (1998), 323–348.

[DR] Z. Ditzian and K. Runovskii, Averages on caps of S^{d-1}, *J. Math. Anal. Appl.* **248** (2000) 260–274.

[R] Kh. P. Rustamov, On the approximation of functions on sphere, *Izv. Akad. Nauk SSSR Ser. Mat.* **59** (1993), 127–148.

[So] C. D. Sogge, Oscillatory integrals and spherical harmonics, *Duke Math. J.* **53** (1986), 43–65.

[St] E. M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1979 (third printing, with corrections).

[SW] E. M. Stein and G Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1971.

[Sz] G. Szegö, "Orthogonal Polynomials," 4th Ed., American Mathematical Society Colloquium Publications 23, Amer. Math. Soc. Providence, RI, 1975.

[WL] Wang Kunyang and Li Luoqing, "Harmonic Analysis and Approximation on the Unit Sphere," Science press, Beijing, 2000.